

Comps Review - Probability Theory

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Four kinds of convergence:

(0) $X_n \rightarrow X$ sample point-wise

(most naïve)

(1) $X_n \xrightarrow{\text{a.s.}} X$

(Strong Law of Large Numbers)

(2) $X_n \xrightarrow{P} X$

(Weak Law of Large Numbers)

(3) $X_n \xrightarrow{L^p} X$

(4) $X_n \xrightarrow{D} X$

(Central Limit Theorem)

DEFINITIONS:

• $X_n \xrightarrow{\text{a.s.}} X$, if $P\{\omega: X_n(\omega) \not\rightarrow X(\omega)\} = 0$;
iff $\forall \varepsilon > 0, P\{|X_n - X| > \varepsilon \text{ i.o.}\} = 0$. (Th 4.2.2)

• $X_n \xrightarrow{P} X$, if $\forall \varepsilon > 0, P\{|X_n - X| < \varepsilon\} \rightarrow 1$, as $n \rightarrow \infty$
Alternative notation: $X_n - X = o_p(1)$.

• $X_n \xrightarrow{L^p} X$, if $\|X_n - X\|_p \rightarrow 0$, as $n \rightarrow \infty$,
i.e. $E|X_n - X|^p \rightarrow 0$, as $n \rightarrow \infty$.

• DEF of L^p -norm: $\|X\|_p := (E|X|^p)^{\frac{1}{p}}$, for $0 < p < \infty$.

• $X_n \xrightarrow{D} X$, if $F_n \Rightarrow F$, i.e. $F_n(x) \rightarrow F(x)$ at all continuous points of F (i.e. when F is cont. at x),
or $X_n \xrightarrow{d} X$,
 $X_n \xrightarrow{d} X$ iff $\mu_n \Rightarrow \mu$, i.e. \exists dense set $D \subset \mathbb{R}^1$, such that
corresponding distribution function
corresponding distribution $\forall a < b \in D, \mu_n(a, b) \rightarrow \mu(a, b)$

• $\mu_n \Rightarrow \mu$ iff $\forall a, b$ such that $\mu\{a\} = \mu\{b\} = 0$,
 $\mu_n(a, b) \rightarrow \mu(a, b)$. (Th 4.3.1)

RELATIONSHIP BETWEEN THEM :

Figure 1.

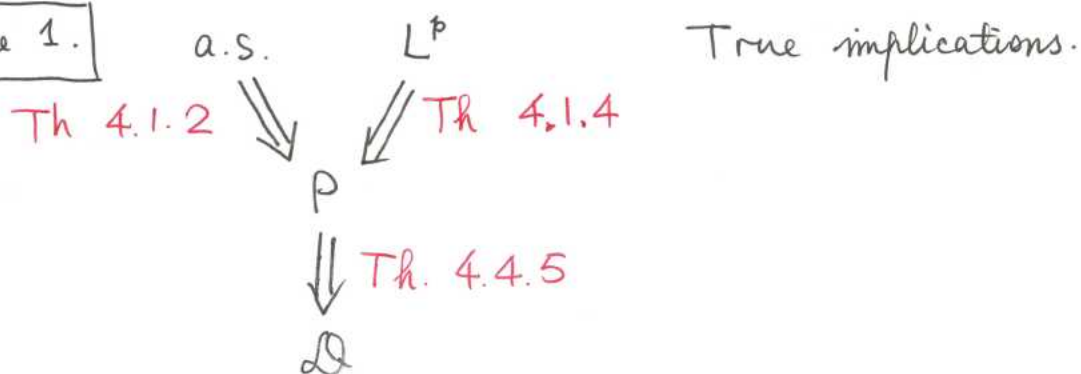
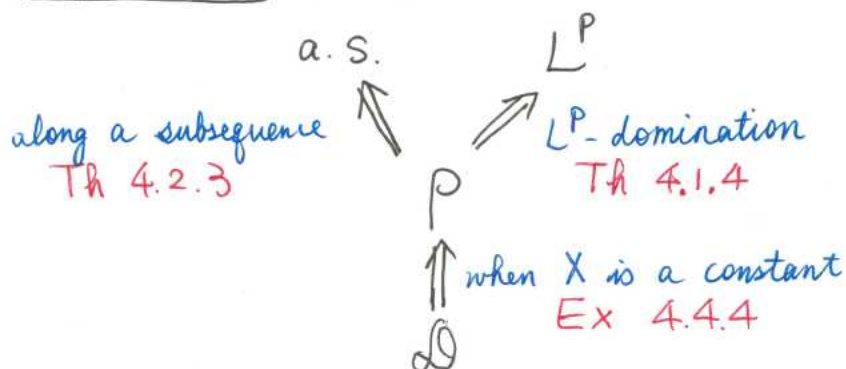


Figure 2. (partial converse - converse ^{holds} with further assumptions)



THM 4.2.3: If $X_n \xrightarrow{P} X$, then $\exists (n_k)$ such that $X_{n_k} \xrightarrow{a.s.} X$ (as $k \rightarrow \infty$).

THM 4.1.4: If $X_n \xrightarrow{P} X$ and $\{X_n\}$ is L^p -dominated, then $X_n \xrightarrow{L^p} X$.
 (only true for $0 < p < \infty$)

• Distinguish between L^p -dominated and L^p -bounded:

- $\{X_n\}$ is L^p -dominated, if \exists random variable $Y \in L^p$, such that $|X_n| \leq Y, \forall n$.
- $\{X_n\}$ is L^p -bounded, if \exists (large but finite) number $M \in \mathbb{R}$, such that $\|X_n\|_p (= [E(X_n)^p]^{1/p}) \leq M, \forall n$.
 (equivalently, if $\sup_n \|X_n\|_p < \infty$).

L^p -domination $\Rightarrow L^p$ -boundedness. But the converse is false.

EX 4.4.4: If $X_n \xrightarrow{\mathcal{Q}} a \in \mathbb{R}$, then $X_n \xrightarrow{P} a$.

Counter examples to Fig 2 without further assumptions.

Example 1 ("circus") $X_n \xrightarrow{L^P} X$, but $X_n \not\xrightarrow{\text{a.s.}} X$;
 $X_n \xrightarrow{P} X$, but $X_n \not\xrightarrow{\text{a.s.}} X$.

$\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B} = \text{Borel on } (0, 1]$, $P = \text{Lebesgue measure on } (0, 1]$.

$$X_0 = I_{(0, 1]},$$

$$X_1 = I_{(0, \frac{1}{2}]} , X_2 = I_{(\frac{1}{2}, 1]},$$

$$X_3 = I_{(0, \frac{1}{3}]} , X_4 = I_{(\frac{1}{3}, \frac{2}{3}]} , X_5 = I_{(\frac{2}{3}, 1]},$$

$$X_6 = I_{(0, \frac{1}{4}]} , X_7 = I_{(\frac{1}{4}, \frac{3}{4}]} , X_8 = I_{(\frac{3}{4}, \frac{3}{4}]} , X_9 = I_{(\frac{3}{4}, 1]},$$

$$\dots$$

$$X_n \xrightarrow{L^P} 0, \quad X_n \xrightarrow{P} 0,$$

But for each sample point, $X_n(\omega)$ has no limit.

Example 2 ("skyscrapers") $X_n \xrightarrow{\text{a.s.}} X$, but $X_n \not\xrightarrow{L^P} X$;
 $X_n \xrightarrow{P} X$, but $X_n \not\xrightarrow{L^P} X$.

$\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}$, $P = \text{Lebesgue measure on } (0, 1]$.

$$X_n = 2^n I_{(0, \frac{1}{n}]}$$

Then $X_n \rightarrow 0$ pointwise

$$\Rightarrow X_n \xrightarrow{\text{a.s.}} 0$$

$$\Rightarrow X_n \xrightarrow{P} 0.$$

But $\|X_n\|_p^p = \frac{2^{np}}{n} \rightarrow \infty$, so $\|X_n\|_p \rightarrow \infty$, so $\{X_n\}$ has no L^p -limit.

Example 3 $X_n \xrightarrow{\mathcal{D}} X$, but $X_n \not\xrightarrow{P} X$

$$X_n, \mu_n := \sum_{k=1}^n \frac{1}{n} \delta_{\frac{k}{n}}, \quad X, \mu: \text{uniform } (0, 1].$$

$$\mu_n \Rightarrow \mu, \quad \text{i.e. } X_n \xrightarrow{\mathcal{D}} X,$$

$$\text{but } P\{|X - X_n| > \frac{1}{2}\} = 1, \quad \forall n. \quad \text{so } X_n \not\xrightarrow{P} X.$$

($\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}$, $P = \text{Lebesgue measure on } (0, 1]$).

IMPORTANT/USEFUL RESULTS

THM (Jensen's inequality) If φ is a convex function, then $\varphi(EX) \leq E[\varphi(X)]$.

THM (Markov's inequality) $\forall a > 0$, $P(|X| \geq a) \leq \frac{E|X|}{a}$.

(useful in controlling tail probability by moments)

• Cor. $P(|X| \geq a) = P(|X|^p \geq a^p) \leq \frac{E|X|^p}{a^p}$.

THM (Slutsky's theorem)

If $X_n \xrightarrow{\mathcal{D}} X$, $Y_n \xrightarrow{\mathcal{D}} c \in \mathbb{R}$, then

- $X_n + Y_n \xrightarrow{\mathcal{D}} X + c$;
- $X_n Y_n \xrightarrow{\mathcal{D}} cX$;
- $Y_n^{-1} X_n \xrightarrow{\mathcal{D}} c^{-1}X$, if $c \neq 0$.

THM 4.5.4 If $0 < r < \infty$, $\forall n: X_n \in L^r$, and $X_n \xrightarrow{P} X$,

then the following are equivalent:

(i) $(|X_n|^r)$ is uniformly integrable;

(ii) $X_n \xrightarrow{L^r} X$

(iii) $\|X_n\|_r \rightarrow \|X\|_r < \infty$.

(relates \xrightarrow{P} , $\xrightarrow{L^r}$, convergence in L^r -norm and u.i.).