A Density w.r.t. a Measure: Normal(0,1)

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Usually, we say a random variable X follows a Normal(0,1) distribution, if its cumulative distribution can be expressed as:

$$P\{X \leq t\} = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

Now we formalize this in a more measure-theoretic way, in correspondence to what we learned in the course, particularly, why the part $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is called the density of X: *How is the term "probability density function" that we use a lot in statistics related to the concept "density" (density of a measure with respect to another measure) that we learned in class?*

First of all, we need to adopt a definition of Normal(0,1) random variable. Say X is a random variable (i.e. measurable function) from $(\Omega, \mathscr{F})$ to $(\mathbb{R}, \mathscr{R})$. Denote P some probability measure on $(\Omega, \mathscr{F})$ and μ the Lebesgue measure on $(\mathbb{R}, \mathscr{R})$. We say X is a Normal(0,1) random variable, if we have (*this is the definition we adopt, i.e. a starting point for the following arguments*)

$$P\{\omega : X(\omega) \leq t\} = \int_{(-\infty, t]} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} d\mu(x). \tag{1}$$

Now, how to convert this into a statement that $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the density of some measure with respect to some other measure? Note that when saying some function D is the density of some measure ρ with respect to some other measure μ, ρ and μ need to be defined on the same measurable space, so at this point we cannot say $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the density of P with respect to μ.

But now the distribution comes to rescue. Recall that at some ealier time point of the class, we've learned the concept "distribution" of a random variable, which is a measure L_X on the target space (here $(\mathbb{R}, \mathscr{R})$) defined as following: for any $A \in \mathscr{R}$,

$$L_X(A) := PX^{-1}(A) = P\{\omega : X(\omega) \in A\}. \tag{2}$$

So by (1) we have

$$L_X((-\infty, t]) = \int_{(-\infty, t]} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} d\mu(x), \tag{3}$$

or (by some careful treatment of the fact that $\mathscr{R}$ is the sigma-field generated by all half-infinity intervals and the properties of measure)

$$L_X(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} d\mu(x), \tag{4}$$

for any $A \in \mathscr{R}$.

That is to say, $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ **is the density of L_X (distribution of X, which is a probability measure) with respect to μ (the Lebesgue measure on the real line).**